

Pointer Quantum PCPs and Multi-Prover Games

Alex B. Grilo^{*1}, Iordanis Kerenidis^{†1,2}, and Attila Pereszlényi^{‡1}

¹IRIF, CNRS, Université Paris Diderot, Paris, France

²Centre for Quantum Technologies, National University of Singapore, Singapore

1st March, 2016

Abstract

The quantum PCP (QPCP) conjecture states that all problems in QMA, the quantum analogue of NP, admit quantum verifiers that only act on a constant number of qubits of a polynomial size quantum proof and have a constant gap between completeness and soundness. Despite an impressive body of work trying to prove or disprove the quantum PCP conjecture, it still remains widely open. The above-mentioned proof verification statement has also been shown equivalent to the QMA-completeness of the Local Hamiltonian problem with constant relative gap. Nevertheless, unlike in the classical case, no equivalent formulation in the language of multi-prover games is known.

In this work, we propose a new type of quantum proof systems, the Pointer QPCP, where a verifier first accesses a classical proof that he can use as a pointer to which qubits from the quantum part of the proof to access. We define the Pointer QPCP conjecture, that states that all problems in QMA admit quantum verifiers that first access a logarithmic number of bits from the classical part of a polynomial size proof, then act on a constant number of qubits from the quantum part of the proof, and have a constant gap between completeness and soundness. We define a new QMA-complete problem, the Set Local Hamiltonian problem, and a new restricted class of quantum multi-prover games, called CRESP games. We use them to provide two other equivalent statements to the Pointer QPCP conjecture: the Set Local Hamiltonian problem with constant relative gap is QMA-complete; and the approximation of the maximum acceptance probability of CRESP games up to a constant additive factor is as hard as QMA. This is the first equivalence between a quantum PCP statement and the inapproximability of quantum multi-prover games.

1 Introduction

The celebrated PCP theorem states that all languages in NP can be verified probabilistically by randomized verifiers that only check a constant number of bits of a polynomial size proof [ALM⁺98, AS98, Din07]. This theorem has far-reaching applications in complexity theory and especially in the inapproximability of certain optimization problems. This is because the PCP theorem can be recast in the following equivalent way: the approximation of MAX-SAT up to some constant additive factor is NP-complete. Let us also remark that the classical PCP theorem has a third very interesting equivalent formulation as approximation of the maximum acceptance probability of some polynomial size multi-prover interactive games [Raz98]. This

^{*}abgrilo@gmail.com

[†]jkeren@gmail.com

[‡]attila.pereszlenyi@gmail.com

game formulation was fundamental in order to achieve better constants for the inapproximability results of a number of NP-hard problems.

One of the main questions in quantum complexity theory is whether one can prove an analogous statement for the class QMA, the quantum analogue of NP. The QPCP conjecture [AAV13] has received a lot of attention due to its importance to both physics and theoretical computer science and an impressive body of work has provided either positive evidence [AALV09, EH15, NV15] or negative [Ara11, AE15, BH13]. There are many different ingredients that go into the proof of the classical PCP theorem, especially since there are two different ways of proving it, one through the proof system formulation and another more combinatorial way by looking directly at the inapproximability of constraint satisfaction problems. In the quantum setting, the positive and negative evidence has been mostly that certain techniques that had been used in the classical setting are applicable or not in the quantum setting. We note that the lack of a way of seeing the quantum PCP conjecture in a game context also prevents us from using some important techniques that are present in the classical case, such as the parallel repetition theorem. Overall, proving the quantum PCP theorem remains a daunting task.

The QPCP conjecture can be cast as a type of proof system which we denote by $\text{QPCP}(q, \alpha, \beta)$. Here a quantum verifier tosses a logarithmic number of classical coins and, based on the coin outcomes, decides on which q qubits from the polynomial-size quantum proof to perform a measurement. The measurement output decides on acceptance or rejection. A yes instance is accepted with probability at least α and a no instance is accepted with probability at most β , for some $\alpha > \beta$ [AALV09, AAV13]. The formal conjecture is stated below.

Conjecture 1.1 (QPCP Conjecture - Proof verification version). $\text{QMA} = \text{QPCP}(q, \alpha, \beta)$ where $q = O(1)$ and $\alpha - \beta = \Omega(1)$.

In quantum mechanics, the evolution of quantum systems are described by Hermitian operators called Hamiltonians. In nature, particles that are far apart tend not to interact so the global Hamiltonian can usually be described as a sum of local Hamiltonians. The Local Hamiltonian problem, denoted by $\text{LOCALHAM}(k, a, b)$, receives as input m Hamiltonians H_1, \dots, H_m where each one describes the evolution of at most k qubits and the question is if there is a global state such that its energy is at most am or all states have energy at least bm for $b > a$. The area studying the above problem is called quantum Hamiltonian complexity [Osb12, GHLS15], a topic that lies in the intersection of physics, computer science and mathematics. It began with Kitaev who showed that for $b - a \geq 1/\text{poly}(n)$, $\text{LOCALHAM}(5, a, b)$ is complete for the class QMA [AN02, KSV02]. It has subsequently been improved, reducing the locality of the Hamiltonians to two [KKR06] and restricting their structure [KR03, OT10, CM14, HNN13]. These results imply that estimating the groundstate energy of a system within an inverse polynomial additive factor is hard. It is natural to ask if it still remains hard if we require only constant approximation. The physical interpretation of this problem is connected to the stability of entanglement in “room temperature”.

The second equivalent statement of the quantum PCP conjecture asks if $\text{LOCALHAM}(k, a, b)$ remains QMA-complete when $b - a$ is constant. It is stated formally in the conjecture below.

Conjecture 1.2 (QPCP Conjecture - Constraint satisfaction version). *The Local Hamiltonian problem $\text{LOCALHAM}(k, a, b)$ is QMA-complete for $k = O(1)$ and $b - a = \Omega(1)$, where the QMA-hardness is with respect to quantum reductions.*

The two versions of the quantum PCP conjecture have been proven equivalent [AAV13], and since Conjecture 1.2 is true for $b - a \geq 1/\text{poly}(n)$, we can also conclude that $\text{QMA} = \text{QPCP}(q, \alpha, \beta)$ with $q = O(1)$ and $\alpha - \beta \geq 1/\text{poly}(n)$.

Let us note that so far there is no equivalent statement of the QPCP conjecture in the language of multi-prover games, though the approximation of the maximum acceptance probability of certain multi-prover games up to an inverse-polynomial additive factor has been proven to be QMA-hard [FV15].

1.1 Our Results

In our work, we propose a new type of quantum proof systems, the Pointer QPCP, and formulate three equivalent versions of the Pointer QPCP conjecture. This may help towards proving or disproving the original QPCP conjecture. We start by describing a new proof verification system then we provide a new variant of the Local Hamiltonian problem and last we describe an equivalent polynomial size multi-prover game. Up to our knowledge, this is the first time a polynomial size multi-prover game has been proven equivalent to some QPCP conjecture. Our new conjecture is a weaker statement than the original QPCP conjecture and hence may be easier to prove. Moreover, having an equivalent game version of it might also lead to new methods that could potentially be relevant for attacking the original conjecture as well.

We now give some details of our results. We define a new quantum proof system, where the proof contains two separate parts, a classical and a quantum proof both of polynomial size. The verifier can first access a logarithmic number of bits from the classical proof and, depending on the content, he can then access a constant number of qubits from the quantum proof. Since the classical part can be seen as a pointer to the qubits that will be accessed, we denote this proof system by $\text{PointerQPCP}(q, \alpha, \beta)$. To be more specific, the verifier first reads a logarithmic number of bits from the classical part of the proof and then measures at most q qubits from the quantum part. He accepts a yes instance with probability at least α and a no instance with probability at most β . Since a Pointer QPCP is a generalization of QPCP, it follows that all problems in QMA have a $\text{PointerQPCP}(q, \alpha, \beta)$ proof system with $\alpha - \beta \geq 1/\text{poly}(n)$.

Conjecture 1.3 (Pointer QPCP Conjecture - Proof verification version). *It holds that $\text{QMA} = \text{PointerQPCP}(q, \alpha, \beta)$ where $q = O(1)$ and $\alpha - \beta = \Omega(1)$.*

We note that quantum proof systems with classical and quantum parts have also appeared in [Raz05]. There, the aim was to reduce the number of blocks being read in classical PCPs and hence, in the proposed model, a logarithmic size quantum proof is provided to the verifier who measures it and then reads only a single block from a polynomial size classical proof.

In addition to Pointer QPCPs, we also propose a “constraint satisfaction” version of the above conjecture which will turn out to be equivalent. We do this by defining a new variant of the Local Hamiltonian problem which we call the Set Local Hamiltonian problem. Here the input is m sets of a polynomial number of k -local Hamiltonians each, and we ask if there exists a representative Hamiltonian from each set such that the Hamiltonian corresponding to their sum has groundstate energy at most am or for every possible choice of representative Hamiltonians from each set, the Hamiltonian corresponding to their sum has groundstate energy at least bm . We denote the above problem by $\text{SLH}(k, a, b)$. Since the Local Hamiltonian problem is a special case of the Set Local Hamiltonian problem, where the sets are singletons, $\text{SLH}(k, a, b)$ is QMA-complete for $k \geq 2$ and $b - a \geq 1/\text{poly}(n)$.

Conjecture 1.4 (Pointer QPCP Conjecture - Constraint satisfaction version). *The $\text{SLH}(k, a, b)$ problem is QMA-complete for $k = O(1)$ and $b - a = \Omega(1)$.*

As mentioned earlier, the classical PCP theorem has another interesting equivalent formulation regarding the approximation of the maximum acceptance probability of multi-prover games [Raz98], while the same is not known for the quantum case. We propose an equivalent multi-prover game formulation of the Pointer QPCP conjecture. Our game, which we call CRES

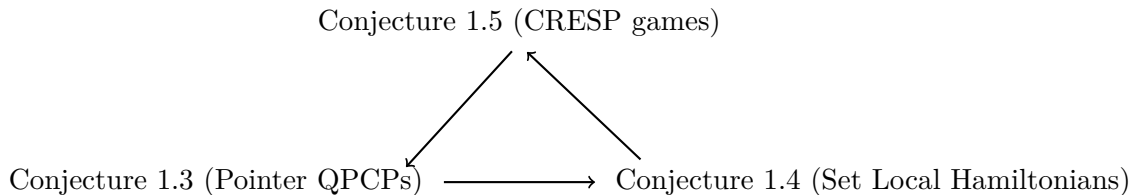


Figure 1: Arrows of implications in our proof of the equivalence of the three conjectures.

(Classical and Restricted-Entanglement Swapping-Provers) game, was inspired by the work of Fitzsimons and Vidick [FV15]. However, in order to prove an equivalence, we had to drastically change the game. In their work, a multi-prover game is proposed for the Local Hamiltonian problem in which the completeness-soundness gap is inverse polynomial. If we try to follow the same proof but with an instance of the Local Hamiltonian with constant gap, the gap does not survive and at the end there will be an inverse-polynomial gap in the game. Hence we are not able to prove the equivalence with the standard QPCP conjecture.

We define our CRESP game to have one classical prover and logarithmically many quantum provers who are restricted both in the strategies they can perform and also in the initial quantum state they share. The verifier asks a single question of logarithmic length to all of them, the classical prover replies with logarithmically many bits, while the quantum provers reply with k 4-dimensional qudits. (For simplicity, we will omit the dimension of the qudit system in the rest of the paper.) The promise problem $\text{CRESP}(k, \alpha, \beta)$ informally asks if we can distinguish between the cases when the provers win the game with probability at least α or at most β . Similarly to the previous problems, we will see that $\text{CRESP}(k, \alpha, \beta)$ is QMA-complete for $\alpha - \beta \geq 1/\text{poly}(n)$. See Theorem 4.6 for the precise statement.

Conjecture 1.5 (Pointer QPCP Conjecture - Game version). *The $\text{CRESP}(k, \alpha, \beta)$ problem is QMA-complete for $k = O(1)$ and $\alpha - \beta = \Omega(1)$.*

Our main result is the equivalence of the above three formulations of the Pointer QPCP conjecture. It is stated formally in the following theorem.

Theorem 1.6 (Main theorem). *The three versions of the Pointer QPCP conjecture (Conjectures 1.3 to 1.5) are either all true or all false.*

The proof is divided into three steps: first, we show that Conjecture 1.3 implies Conjecture 1.4; second, we show that Conjecture 1.4 implies Conjecture 1.5; and finally, we prove that Conjecture 1.5 implies Conjecture 1.3. The arrows of implications are depicted in Fig. 1.

The paper is organized as follows: In Section 2, we describe some standard definitions required for the rest of the paper. In Section 3, we present the definitions of our new notions, the Pointer QPCPs, the Set Local Hamiltonian problem, and the CRESP games. The proof of equivalence is presented in Section 4. We conclude the paper with some discussion and open problems in Section 5.

2 Preliminaries

In this section we provide some definitions that we use in the paper. We start by defining QMA, the quantum analogue of NP.

Definition 2.1 (Quantum Merlin-Arthur proof systems). *Let $n \in \mathbb{Z}^+$ be the input size and p be a polynomial. A QMA protocol proceeds in the following steps.*

1. The verifier receives an input x and a quantum proof $|\psi\rangle$ of size $p(n)$.
2. The verifier runs in polynomial time in n . He performs a general POVM measurement on $|\psi\rangle$ and decides on the acceptance or rejection of the input.

A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ belongs to QMA if it has a QMA proof system with the following properties.

Completeness. If $x \in A_{\text{yes}}$ then there is a $|\psi\rangle$ such that the verifier accepts with probability at least $\frac{2}{3}$.

Soundness. If $x \in A_{\text{no}}$ then for all $|\psi\rangle$ the verifier accepts with probability at most $\frac{1}{3}$.

Now we present the Local Hamiltonian problem, the quantum analogue of MAX-SAT.

Definition 2.2. The Local Hamiltonian problem is denoted by $\text{LOCALHAM}(k, a, b)$ where $k \in \mathbb{Z}^+$ is called the locality and for $a, b \in \mathbb{R}$ it holds that $a < b$. It is the following promise problem. Let n be the number of qubits of a quantum system. The input is a set of $m(n)$ Hamiltonians $H_1, \dots, H_{m(n)}$ where m is a polynomial in n , $\forall i \in [m(n)] : 0 \leq H_i \leq \mathbb{1}$ and each H_i acts on k qubits out of the n qubit system. For $H \stackrel{\text{def}}{=} \sum_{j=1}^{m(n)} H_j$ the following two conditions hold.

- In a YES instance there exists a state $|\varphi\rangle \in \mathbb{C}^{2^n}$ such that $\langle \varphi | H | \varphi \rangle \leq a \cdot m(n)$.
- In a NO instance for all states $|\varphi\rangle \in \mathbb{C}^{2^n}$ it holds that $\langle \varphi | H | \varphi \rangle \geq b \cdot m(n)$.

Kitaev proved that for $k \geq 5$ and $b - a \geq 1/\text{poly}(n)$ the $\text{LOCALHAM}(k, a, b)$ problem is QMA-complete [KSV02]. This completeness result was later improved for $k \geq 2$ [KKR06].

We now define the quantum analogue of PCPs, a quantum proof system where the verifier only checks a few qubits from the quantum proof.

Definition 2.3 (Quantum Probabilistically Checkable Proofs). Let $n \in \mathbb{Z}^+$ be the input size and p be a polynomial. A QPCP protocol proceeds in the following steps.

1. The verifier receives an input x and a quantum proof $|\psi\rangle$ of size $p(n)$.
2. The verifier runs in time polynomial in n . He picks $O(\log n)$ bits uniformly at random, and based on the input and on the random bits, he performs a general POVM measurement on q qubits, and decides on acceptance or rejection of the input.

A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ belongs to $\text{QPCP}(q, \alpha, \beta)$ if it has a QPCP proof system with the following properties.

Completeness. If $x \in A_{\text{yes}}$ then there is a $|\psi\rangle$ such that the verifier accepts with probability at least α .

Soundness. If $x \in A_{\text{no}}$ then for all $|\psi\rangle$ the verifier accepts with probability at most β .

We can easily prove the following statement.

Lemma 2.4. It holds that $\text{QMA} = \text{QPCP}(q, \alpha, \beta)$ where $q = O(1)$ and $\alpha - \beta \geq 1/\text{poly}(n)$.

Proof. The containment $\text{QPCP}(q, \alpha, \beta) \subseteq \text{QMA}$ is trivial since the QMA verifier can read the whole proof and the power of QMA doesn't change if the gap is inverse-polynomial. The other direction of the containment follows from Kitaev's proof that the 5-Local Hamiltonian is QMA-complete [KSV02, AN02]. The QMA verifier in the proof is also a QPCP verifier. \square

The quantum PCP conjecture has two equivalent versions:

Theorem 2.5 ([AAV13]). *The class QMA is equal to the class QPCP(q, α, β) with $q = O(1)$ and $\alpha - \beta = \Omega(1)$ (Conjecture 1.1) if and only if the Local Hamiltonian problem LOCALHAM(k, a, b) is QMA-complete for $k = O(1)$ and $b - a = \Omega(1)$, where the QMA-hardness is with respect to quantum reductions (Conjecture 1.2).*

3 Pointer QPCPs, Set Local Hamiltonians, and CRESP Games

In this section, we present the definitions required for our conjectures. We start by defining Pointer QPCPs, a generalized version of QPCPs, in which the verifier can read a small number of bits from the classical part of the proof and then, based on that, read a constant number of qubits from the quantum part of the proof. Then we propose the Set Local Hamiltonian problem that can be thought of as a “constraint satisfaction” version of the conjecture. Finally, we define CRESP games which are restricted multi-prover games for which approximation of their value will turn out to be equivalent to the other two formulations.

3.1 Pointer QPCPs

Definition 3.1. *Let $n \in \mathbb{Z}^+$ be the input size and let m, l, p be polynomials. A Pointer QPCP protocol proceeds in the following steps.*

1. *The verifier receives an input x and a two-part proof of size $m(n) + p(n)$ in the form*

$$y_1 \dots y_{m(n)} \otimes |\psi\rangle$$

where $y_i \in [l(n)]$ (i.e., each y_i can be written with $O(\log n)$ bits) and $|\psi\rangle$ is a state of $p(n)$ qubits. We refer to $y_1 \dots y_{m(n)}$ as the classical part of the proof and $|\psi\rangle$ as the quantum part of the proof.

2. *The verifier runs in time polynomial in n . He chooses uniformly at random a position $i \in [m(n)]$ of the classical proof to read. Then, based on his input, the random bits and the value of y_i , he chooses q qubits from the quantum proof, performs a general POVM measurement on them, and decides on acceptance or rejection of the input.*

A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ belongs to PointerQPCP(q, α, β) if it has a Pointer QPCP proof system with the following properties.

Completeness. *If $x \in A_{\text{yes}}$ then there exists $y_1 \dots y_{m(n)} \otimes |\psi\rangle$ such that verifier accepts with probability at least α .*

Soundness. *If $x \in A_{\text{no}}$ then for all $y_1 \dots y_{m(n)} \otimes |\psi\rangle$ the verifier accepts with probability at most β .*

Lemma 3.2. *QMA = PointerQPCP(q, α, β) where $q = O(1)$ and $\alpha - \beta \geq 1/\text{poly}(n)$.*

Proof. Since Pointer QPCPs are generalizations of QPCPs, we have that $\text{QPCP}(q, \alpha, \beta) \subseteq \text{PointerQPCP}(q, \alpha, \beta)$ for any values of q, α , and β . From Lemma 2.4, it follows that $\text{QMA} \subseteq \text{PointerQPCP}(q, \alpha, \beta)$. The other direction of the containment follows trivially since the QMA verifier can read the whole proof. \square

Conjecture 1.3 asks whether the class QMA also has Pointer QPCPs with $q = O(1)$ and $\alpha - \beta = \Omega(1)$.

3.2 The Set Local Hamiltonian Problem

We define a new QMA-complete problem which is a generalization of the Local Hamiltonian problem and which will lead to another version of our conjecture.

Definition 3.3 (Set Local Hamiltonian Problem). *The Set Local Hamiltonian problem is denoted by $\text{SLH}(k, a, b)$ where $k \in \mathbb{Z}^+$ is called the locality and for $a, b \in \mathbb{R}$ it holds that $a < b$. It is the following promise problem. Let n be the number of the qubits of a quantum system, and m and l be two polynomials. The input for the problem are $m(n)$ sets of Hamiltonians. For all $i \in [m(n)]$ the set \mathbf{H}_i contains $l(n)$ Hamiltonians, i.e.,*

$$\forall i \in [m(n)] : \mathbf{H}_i = \{H_{i,1}, \dots, H_{i,l(n)}\}.$$

Each Hamiltonian is positive and has norm at most one, i.e., $\forall i \in [m(n)], \forall j \in [l(n)] : 0 \leq H_{i,j} \leq \mathbf{1}$. Each Hamiltonian acts non-trivially on at most k qubits out of the n qubits of the quantum system. The problem is to decide which one of the following two conditions hold.

- *In a YES instance, there exists a function $f : [m(n)] \rightarrow [l(n)]$ and a state $|\varphi\rangle \in \mathbb{C}^{2^n}$ such that*

$$\langle \varphi | \sum_{i=1}^{m(n)} H_{i,f(i)} | \varphi \rangle \leq a \cdot m(n).$$

- *In a NO instance, for all functions $f : [m(n)] \rightarrow [l(n)]$ and for all states $|\varphi\rangle \in \mathbb{C}^{2^n}$, we have that*

$$\langle \varphi | \sum_{i=1}^{m(n)} H_{i,f(i)} | \varphi \rangle \geq b \cdot m(n).$$

Lemma 3.4. *The $\text{SLH}(k, a, b)$ problem is QMA-complete for $k \geq 2$ and $b - a \geq 1/\text{poly}(n)$.*

Proof. For the containment $\text{SLH}(k, a, b) \in \text{QMA}$, let the witness have a classical part that contains the description of the function f and a quantum part that is supposed to be the state $|\varphi\rangle$. The quantum verifier can then apply the usual eigenvalue estimation on $\sum_{i=1}^{m(n)} H_{i,f(i)}$. The hardness of $\text{SLH}(k, a, b)$ comes trivially from the fact that Local Hamiltonian problem is a special case of the Set Local Hamiltonian problem with $l(n) = 1$. \square

Note that Conjecture 1.4 asks whether the Set Local Hamiltonian problem remains QMA-complete when the locality is constant and the gap between b and a is also constant.

3.3 CRESP Games

We now formally describe a new variant of quantum multi-prover games. These games are rather restricted but will allow us to state a third variant of our pointer QPCP conjecture.

3.3.1 Description of the Game

Let $n \in \mathbb{Z}^+$ be a parameter and m be a polynomial. The size of the game will be polynomial in n . The game is played by one classical prover, $\lceil \log(n+1) \rceil$ quantum provers, and a verifier. It is played as follows.

1. The quantum provers share the encoding of an arbitrary n -qubit state. (The encoding maps each qubit into a number of qudits and will be defined later.) They are not allowed to share any other resources.

2. The verifier picks a question i uniformly at random out of the $m(n)$ possible questions and sends the same question to all the provers (both quantum and classical).
3. The classical prover replies with $O(\log n)$ bits.
4. Each quantum prover replies by at most k qudits from their shared encoded state. All the quantum provers use the same strategy.
5. The verifier accepts or rejects, based on his question and the answers from the provers.

We denote these games by the acronym CRESP after the **C**lassical prover, the **R**estricted **E**ntanglement that the quantum provers can share and, since the only possible strategy the quantum provers can perform is to swap some of their qudits into the message register, we call them **S**wapping-**P**rovers.

3.3.2 Restriction on the Entanglement

The entangled state the provers share is of the following predefined form. First, the provers pick an arbitrary n -qubit state $|\phi\rangle \in \mathbb{C}^{2^n}$. The state $|\phi\rangle$ is encoded with a linear isometry $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \dots \otimes \mathcal{E}_n$ where each qubit of $|\phi\rangle$ is encoded with $\mathcal{E}_i : \mathbb{C}^2 \rightarrow \bigotimes_{j=1}^{\lceil \log(n+1) \rceil} \mathcal{H}_{i,j}$. For all i and j , $\mathcal{H}_{i,j} \cong \mathbb{C}^4$, that is, $\mathcal{H}_{i,j}$ is a four-dimensional space which we simply call qudit. To define \mathcal{E}_i , let's fix some ordering on the non-empty subsets of $[\lceil \log(n+1) \rceil]$. Let Q_i be the i -th subset, $\mathcal{S}_i \stackrel{\text{def}}{=} \bigotimes_{j \in Q_i} \mathcal{H}_{i,j}$, and $\overline{\mathcal{S}}_i \stackrel{\text{def}}{=} \bigotimes_{j \notin Q_i} \mathcal{H}_{i,j}$. For each $i \in [n]$, we define \mathcal{E}_i by giving its action on the standard basis states.

$$\mathcal{E}_i(|0\rangle) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes |Q_i|} + |1\rangle^{\otimes |Q_i|} \right)_{\mathcal{S}_i} \otimes \left(|0\rangle^{\otimes \lceil \log(n+1) \rceil - |Q_i|} \right)_{\overline{\mathcal{S}}_i} \quad (1)$$

$$\mathcal{E}_i(|1\rangle) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left(|2\rangle^{\otimes |Q_i|} + |3\rangle^{\otimes |Q_i|} \right)_{\mathcal{S}_i} \otimes \left(|0\rangle^{\otimes \lceil \log(n+1) \rceil - |Q_i|} \right)_{\overline{\mathcal{S}}_i} \quad (2)$$

We refer to the states in \mathcal{S}_i as GHZ-like states. After \mathcal{E} is applied, prover j receives the qudits that live in space $\bigotimes_{i=1}^n \mathcal{H}_{i,j}$. A possible distribution of the qudits is depicted in Fig. 2.

3.3.3 Description of the CRESP Problem

We are interested in the maximum acceptance probability the provers can achieve, which is called the value of the game. Here the maximum is taken over all legitimate shared states and all legitimate provers' strategies. We now define the promise problem that corresponds to the approximation of the value of CRESP games.

	qubit 1	qubit 2	qubit 3	qubit 4	qubit 5	qubit 6	qubit 7
prover 1							
prover 2							
prover 3							

Figure 2: A possible distribution of the encoding of 7 qubits among 3 provers. The red cells correspond to the GHZ-like entangled states, while the white cells to $|0\rangle$ states.

Definition 3.5. Let $k \in \mathbb{Z}^+$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha > \beta$. Then, $\text{CRESP}(k, \alpha, \beta)$ is the following promise problem. The input is the description of a CRESP game defined above where the quantum provers answer at most k qudits and the following conditions hold.

- In a YES instance the value of the game is at least α .
- In a NO instance the value of the game is at most β .

We will prove that the $\text{CRESP}(k, \alpha, \beta)$ problem is QMA-complete for $k = O(1)$ and $\alpha - \beta \geq 1/\text{poly}(n)$. We defer this proof to Section 4.2 as it needs results that we will establish later. Again, we note that Conjecture 1.5 asks whether $\text{CRESP}(k, \alpha, \beta)$ remains QMA-complete when $k = O(1)$ and $\alpha - \beta = \Omega(1)$.

4 Equivalence of Our QPCP Conjectures

In this section we prove Theorem 1.6, the equivalence of the three versions of our Pointer QPCP conjecture. The proof proceeds in the following three steps. In Section 4.1, we show that if Conjecture 1.3 is true then Conjecture 1.4 is also true. We do this by reducing any problem with a Pointer QPCP proof system to the Set Local Hamiltonian problem. In Section 4.2 we show that if Conjecture 1.4 is true then Conjecture 1.5 is also true by giving a reduction from the Set Local Hamiltonian problem to our decision problem involving CRESP games. To complete the cycle, we prove in Section 4.3 that if Conjecture 1.5 is true then Conjecture 1.3 is also true by giving a Pointer QPCP proof system for an arbitrary CRESP game. See Fig. 1 for the arrows of implications.

4.1 From Pointer QPCP to the Set Local Hamiltonian Problem

In this section we show that if Conjecture 1.3 is true then Conjecture 1.4 is also true. We show that any problem $P \in \text{PointerQPCP}(q, \alpha, \beta)$ is polynomial-time reducible to $\text{SLH}(q, 1 - \alpha, 1 - \beta)$. Assuming Conjecture 1.3, this means that the Set Local Hamiltonian problem is QMA-hard. The containment of the Set Local Hamiltonian problem in QMA is implied by Lemma 3.4.

Theorem 4.1. Any problem $P \in \text{PointerQPCP}(q, \alpha, \beta)$ can be reduced to $\text{SLH}(q, 1 - \alpha, 1 - \beta)$ in polynomial time.

Proof. Let y_1, \dots, y_m be the classical part of the proof where $m = m(n)$ for a polynomial m and $|\psi\rangle$ be the quantum part of the proof, which contains $p(n)$ qubits, for a polynomial p . Suppose that each y_i can take $l = l(n)$ different values, for a polynomial l . We construct an instance of the Set Local Hamiltonian problem that consists of m sets of Hamiltonians \mathbf{H}_i , for $i \in [m]$, where $\mathbf{H}_i = \{H_{i,j}\}_{j \in [l]}$ and the Hamiltonians act on a $p(n)$ -qubit system. Let $H_{i,j}$ be the rejection POVM element of the Pointer QPCP verifier over the constant number of qubits when he reads register i from the classical part of the proof and it contains the value j , i.e., $j = y_i$.

First we prove that if there is a proof that makes the Pointer QPCP verifier accept with probability greater than α then there is a function f such that the groundstate of $\sum_{i=1}^m H_{i,f(i)}$ has energy at most $(1 - \alpha)m$. Let $y_1 \dots y_m \otimes |\psi\rangle$ be such proof and let α_i be the acceptance probability of the Pointer QPCP verifier when the verifier queries i . Since the verifier picks an i uniformly at random, it follows that $\frac{1}{m} \sum_i \alpha_i = \alpha$. Let $f(i) \stackrel{\text{def}}{=} y_i$. In this case, the energy of $|\psi\rangle$ on $\sum_i H_{i,f(i)}$ is

$$\langle \psi | \left(\sum_i H_{i,y_i} \right) | \psi \rangle \leq \sum_i (1 - \alpha_i) = (1 - \alpha)m.$$

For the other direction of the proof, suppose that there is a function f and a state $|\psi\rangle$ such that $\langle\psi| \left(\sum_i H_{i,f(i)} \right) |\psi\rangle \leq (1 - \beta)m$. Then there is a proof that makes the Pointer QPCP verifier accept with probability bigger than β . Let $(f(1), f(2), \dots, f(m)) \otimes |\psi\rangle$ be the proof for the Pointer QPCP verifier. The acceptance probability of the Pointer QPCP verifier with this proof is

$$\frac{1}{m} \sum_i (1 - \langle\psi| H_{i,f(i)} |\psi\rangle) = 1 - \frac{1}{m} \langle\psi| \left(\sum_i H_{i,f(i)} \right) |\psi\rangle \geq \beta.$$

This finishes the proof of the reduction. \square

4.2 From the Set Local Hamiltonian Problem to CRESP Games

In this section we show that if Conjecture 1.4 is true then Conjecture 1.5 is also true. We do this by giving a reduction from the SLH(k, a, b) problem to the CRESP($k, 1 - a/2, 1 - b/2$) problem. Assuming Conjecture 1.4, this implies that the CRESP($k, 1 - a/2, 1 - b/2$) problem is QMA-hard. We prove the containment $\text{CRESP}(k, 1 - a/2, 1 - b/2) \in \text{QMA}$ in Theorem 4.6.

We construct a CRESP game for the Set Local Hamiltonian problem. The main idea in the construction is the following. In our game, the verifier picks an index $i \in [m]$ uniformly at random and sends i to all the provers. The classical prover tells the verifier the specific Hamiltonian that should be taken from set i , i.e., the value of $f(i)$. The quantum provers share the encoded groundstate of the Hamiltonian $\sum_i H_{i,f(i)}$ and reply with the encoding of the qubits that are involved in Hamiltonian $H_{i,f(i)}$.

First, the verifier checks if the received qudits lie in the codespace of qubit i , and if not he rejects. Using the definition of the encoding, the projector onto the codespace is described by

$$(\Pi_i)_{\mathcal{S}_i} \otimes |0\rangle\langle 0|_{\overline{\mathcal{S}_i}}$$

where

$$\Pi_i = \frac{1}{2} \left(\sum_{u,v \in \{0,1\}} |u^{Q_i}\rangle \langle v^{Q_i}| + \sum_{w,z \in \{2,3\}} |w^{Q_i}\rangle \langle z^{Q_i}| \right).$$

Actually, we will see in Lemma 4.4 that it suffices for the verifier to perform only projection $(\Pi_i)_{\mathcal{S}_i}$. If the above test succeeds then the verifier picks a bit uniformly at random and if it is 0, he accepts. Otherwise, the verifier decodes the answered qudits by inverting the mapping \mathcal{E} , defined by Eqs. (1) and (2), for all the qubits in Hamiltonian $H_{i,f(i)}$. Then, he performs the measurement that corresponds to $H_{i,f(i)}$ on the decoded qubits and accepts or rejects based on the outcome.

If the Hamiltonian $\sum_i H_{i,f(i)}$ has an eigenstate with small eigenvalue then the provers will pass the test with high probability. Using the fact that the provers share a state in the predefined encoding and the restriction on the quantum provers' strategies, we also show that the verifier will reject with high probability if all states have high eigenvalues. The description of the game is in Item 5.

Theorem 4.2. *The game defined by Item 5 has completeness $1 - a/2$ and soundness $1 - b/2$.*

Proof. Lemma 4.3 proves completeness while Lemma 4.5 proves soundness. \square

Lemma 4.3 (Completeness). *If there is a function f such that the groundstate of $\sum_i H_{i,f(i)}$ has eigenvalue at most am then the maximum acceptance probability of the game is at least $1 - a/2$.*

Protocol 1 CRESP Game for $\text{SLH}(k, a, b)$

1. The provers pick an n -qubit state $|\phi\rangle$ and share its encoding $\mathcal{E}(|\phi\rangle)$. (In the honest case, $|\phi\rangle$ is supposed to be the groundstate of Hamiltonian H .)
 2. The verifier picks $i \in [m]$ uniformly at random and sends it to all the provers.
 3. The classical prover sends some $j \in [l]$.
 4. Each quantum provers send k qudits.
 5. The verifier performs the following tests.
 - T_1 . Measure the received qudits that correspond to the space \mathcal{S}_i with the projectors $\{\Pi_i, \Pi_i^\perp\}$ and reject if the outcome is Π_i^\perp . Otherwise, continue.
 - T_2 . Pick $b \in \{0, 1\}$ uniformly at random and accept if $b = 0$. Otherwise, continue.
 - T_3 . Decode the received qudits and perform the measurement specified by $H_{i,f(i)}$ and accept or reject depending on the outcome.
-

Proof. Let the quantum provers share $\mathcal{E}(|\psi\rangle)$, the encoding of the groundstate $|\psi\rangle$ of $H \stackrel{\text{def}}{=} \sum_i H_{i,f(i)}$. When the verifier queries i , the classical prover answers $f(i)$ and all quantum provers honestly reply with their shares of the encodings of the k qubits corresponding to $H_{i,f(i)}$. The verifier always measures Π_i and so he accepts with probability

$$\frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{m} \sum_{i=1}^m \langle \psi | H_{i,f(i)} | \psi \rangle \right) = 1 - \frac{1}{2m} \langle \psi | H | \psi \rangle \geq 1 - \frac{a}{2}. \quad \square$$

The following technical lemma is the key to prove soundness. It establishes that when the provers reply with the qudits that belong to the encoding of a different qubit, the verifier will detect it with probability at least half. We defer the full proof of this lemma to Appendix A.

Lemma 4.4. *If the provers are asked for the encoding of qubit i and they answer with the qudits that correspond to the encoding of a different qubit, then the answered state projects to the correct codespace, i.e., the subspace that corresponds to the projector Π_i , with probability at most $1/2$.*

Proof Sketch. Since the provers have the same strategy for a fixed question and they can only do swaps, the only cheating strategy for the provers is to answer the encoding of a qubit which is different from the one the verifier asked for. In this case, by the properties of the chosen encoding, the state that should be a GHZ-like state is actually separable and it projects to the correct codespace with probability at most half. \square

We prove now the soundness property of the game.

Lemma 4.5 (Soundness). *If, for all functions f , the groundstate of $\sum_i H_{i,f(i)}$ has eigenvalue at least bm then the maximum acceptance probability of the game is at most $1 - b/2$.*

Proof. Let's fix an arbitrary strategy for the classical prover. We define the function f by letting $f(i)$ be the answer from the classical prover to question i . Let us also fix the shared state of the quantum provers arbitrarily. The strategy of the quantum provers is still left undefined.

Let A be the event that the verifier accepts, E_i be the event that the verifier picks question i , T_1 be the event that the verifier continues after the test T_1 , i.e., the answered qudits were projected onto the codespace, T_2 be the event that the verifier accepts during test T_2 , i.e., he

picks $b = 0$, and T_3 be the event that the verifier accepts on test T_3 , i.e., in the estimation of the energy of the Hamiltonian. Let us fix a strategy where the quantum provers answer honestly, i.e., where $\Pr[T_1|E_i] = 1$. The probability of acceptance with this strategy is

$$\begin{aligned}\Pr[A] &= \sum_{i=1}^m \Pr[E_i] \cdot \Pr[T_1|E_i] \left(\Pr[T_2] + \Pr[\bar{T}_2] \cdot \Pr[T_3|E_i, T_1, \bar{T}_2] \right) \\ &= \sum_{i=1}^m \frac{1}{m} \left(\frac{1}{2} + \frac{1}{2} \cdot \Pr[T_3|E_i, T_1, \bar{T}_2] \right) \\ &\leq \frac{1}{2} + \frac{1}{2} (1 - b) \\ &= 1 - \frac{b}{2}\end{aligned}$$

where the inequality follows from the fact that all the eigenstates of $\sum_i H_{i,f(i)}$ have eigenvalue at least bm .

Let G and B be two strategies which are the same except when question r is asked for a fixed r . When asked r , the provers following G answer honestly while provers following B answer with the encoding of a different qubit.

We extend the previous notation by adding the superscript of the corresponding strategy for the events, e.g., T_1^G is the event that the answered qudits are in the correct codespace when the provers follow strategy G . The following calculation shows that G has higher success probability.

$$\begin{aligned}\Pr[A^G] - \Pr[A^B] &= \Pr[E_k^G] \cdot \Pr[T_1^G|E_k^G] \left(\Pr[T_2^G] + \Pr[\bar{T}_2^G] \cdot \Pr[T_3^G|E_k^G, T_1^G, \bar{T}_2^G] \right) \\ &\quad - \Pr[E_k^B] \cdot \Pr[T_1^B|E_k^B] \left(\Pr[T_2^B] + \Pr[\bar{T}_2^B] \cdot \Pr[T_3^B|E_k^B, T_1^B, \bar{T}_2^B] \right) \\ &\geq \frac{1}{m} \left(1 \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \Pr[T_3^G|E_k^G, T_1^G, \bar{T}_2^G] \right) \right. \\ &\quad \left. - \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2} \cdot \Pr[T_3^B|E_k^B, T_1^B, \bar{T}_2^B] \right) \right) \\ &\geq \frac{1}{m} \left(\frac{1}{2} - \frac{1}{2} \right) \\ &= 0\end{aligned}$$

where in the first inequality we used that $\Pr[T_2^B|E_k^B] \leq 1/2$, by Lemma 4.4. By a hybrid argument, it is easy to see that the strategy where provers are honest is optimal and so the soundness follows. \square

We now show that even though our game seems very restricted, it is in fact QMA-hard to approximate its value to within an inverse-polynomial precision.

Theorem 4.6. *The CRESP(k, α, β) problem is QMA-complete for $k = O(1)$ and $\alpha - \beta \geq 1/\text{poly}(n)$.*

Proof. The containment in QMA is simple: The QMA proof is the state the provers choose before the encoding together with the classical information that describes the behavior of all the provers. Then the QMA verifier can create the encoding and simulate the game. This leads to the same acceptance probability as that of the game which means that there is an inverse-polynomial gap between completeness and soundness in the QMA protocol.

The QMA-hardness follows from Lemma 3.4 and Theorem 4.2. \square

4.3 From CRESP Games to Pointer QPCPs

In this section we show that if Conjecture 1.5 is true then Conjecture 1.3 is also true. We do this by proving that $\text{CRESP}(k, \alpha, \beta) \in \text{PointerQPCP}(k, \alpha, \beta)$. Assuming Conjecture 1.5, this implies that $\text{QMA} \subseteq \text{PointerQPCP}(k, \alpha, \beta)$. The inclusion $\text{PointerQPCP}(k, \alpha, \beta) \subseteq \text{QMA}$ follows trivially, the same way as in Lemma 3.2.

Theorem 4.7. $\text{CRESP}(k, \alpha, \beta) \in \text{PointerQPCP}(k, \alpha, \beta)$.

Proof. In CRESP games, the strategy of the quantum provers consists of the choice of the shared state and the choice of which qudits to answer for each one of the verifier's questions. For the classical prover, the strategy consists of the classical answers for each one of the verifier's questions. Therefore, we can have a Pointer QPCP whose proof will be as follows: for the classical part, for each possible question of the verifier, we include the indices of the qudits answered by the quantum provers and the answer of the classical prover. The quantum part of the proof will be the shared state before the encoding. With this information, the verifier of the Pointer QPCP can simulate the classical prover, the quantum provers, and the verifier of the CRESP game.

Formally, the verifier of the Pointer QPCP protocol is provided a proof of the form $y_1 \dots y_m \otimes |\psi\rangle$, where y_i can be seen as a pair (s_i, c_i) . The verifier will do the following.

1. He picks a question i uniformly at random as the verifier of the game.
2. He reads the corresponding strategy of the provers, i.e., (s_i, c_i) .
3. He creates the encoding of the qubits that are specified by strategy s_i .
4. He simulates the verifier of the game using the encoded qubits as the quantum provers' answers and c_i as the classical prover's answer.
5. He accepts if and only if the verifier of the game accepts.

In our construction, we crucially use the fact that each quantum prover has the same strategy, as otherwise, the QPCP verifier would need to read out the strategies of each prover, which would require $\Omega(\log^2(n))$ bits of information. Note that we only read out k qubits from the quantum part of the proof. We are left to prove completeness and soundness.

For completeness, it is not hard to see that if there is a strategy for the provers in the game with acceptance probability p then there is a Pointer QPCP that accepts with probability p as well, just by providing the values of s_i , c_i , and $|\psi\rangle$ that lead to acceptance with probability p in the game.

For soundness, if there are values of $y_i = (s_i, c_i)$ and $|\psi\rangle$ that make the Pointer QPCP verifier accept with some probability then these values can be translated to strategies of the provers in the CRESP game that will achieve the same acceptance probability. \square

5 Discussions and Open Problems

We defined a new variant of quantum proof systems, the Pointer QPCPs, and provided three equivalent versions of the Pointer QPCP conjecture. Our conjecture is weaker than the original QPCP conjecture and hence may be easier to prove. Moreover, the fact of having an equivalent game formulation might lead to new techniques for resolving the conjecture.

It is an interesting question to see whether we can define a more natural game which is equivalent to the Pointer QPCP conjecture. For our equivalence, we were forced to impose

stringent constraints on the game. Nevertheless, it seems that if we allow the quantum provers to either share some more general entangled state or apply any operator to the state they share other than swapping, then it is not clear how not to lose the constant gap when constructing the witness [FV15, Ji15] or not to increase the question size to exponential [NV15]. To illustrate this problem, imagine that the provers are allowed to slightly change the states they return depending on the questions that were asked. If the amount of change, in the trace distance, is in the order of $o(1)$ then the verifier will not be able to detect this with constant probability.¹ When we try to prove soundness by constructing a state with low energy (as in [FV15]) these errors add up $\Omega(n)$ times. The final error can then be too big so we don't know whether the constructed state is a yes or no instance.

Even with our constraints, CRESP games remain equivalent to Pointer QPCPs. Since Pointer QPCPs are a superclass of QPCPs, finding a game that is equivalent to the original QPCP would potentially impose even further constraints. One of the main problems going from a game back to Local Hamiltonians or QPCPs, is that to simulate the game, the strategies of the provers must be somehow simulated and when we try to do this with Local Hamiltonians, the gap vanishes, while for QPCPs, we require the classical pointer queries. Note that, in the Set Local Hamiltonian problem the gap doesn't depend on the size of the sets, by definition. Whereas, if we want to go to the usual Local Hamiltonian problem then the absolute gap is divided by the total number of Hamiltonians and so the gap vanishes.

Acknowledgements

We thank Thomas Vidick for helpful discussions. Supported by ERC QCC, ANR RDAM.

References

- [AALV09] Dorit Aharonov, Itai Arad, Zeph Landau, and Umesh Vazirani. The detectability lemma and quantum gap amplification. In *Proceedings of the 41st Annual ACM Symposium on Theory of Computing (STOC '09)*, pages 417–426, 2009.
- [AAV13] Dorit Aharonov, Itai Arad, and Thomas Vidick. Guest column: the quantum PCP conjecture. *SIGACT News*, 44(2):47–79, 2013.
- [AE15] Dorit Aharonov and Lior Eldar. The commuting local Hamiltonian problem on locally expanding graphs is approximable in NP. *Quantum Information Processing*, 14(1):83–101, January 2015.
- [ALM⁺98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, May 1998.
- [AN02] Dorit Aharonov and Tomer Naveh. Quantum NP - A Survey, 2002, arXiv:quantum-ph/0210077.
- [Ara11] Itai Arad. A Note About a Partial No-go Theorem for Quantum PCP. *Quantum Info. Comput.*, 11(11-12):1019–1027, November 2011.
- [AS98] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, January 1998.

¹Let the amount of change be in the order of $\Theta(1/\sqrt{n})$, for example.

- [BH13] Fernando G. S. L. Brandão and Aram W. Harrow. Product-state approximations to quantum ground states. In *Proceedings of the 45th Annual ACM Symposium on Theory of Computing (STOC '13)*, pages 871–880, 2013.
- [CM14] Toby S. Cubitt and Ashley Montanaro. Complexity classification of local Hamiltonian problems. In *Proceedings of the 55th IEEE Annual Symposium on Foundations of Computer Science (FOCS '14)*, pages 120–129, 2014.
- [Din07] Irit Dinur. The PCP Theorem by Gap Amplification. *Journal of the ACM*, 54(3), June 2007.
- [EH15] Lior Eldar and Aram W. Harrow. Local Hamiltonians with no low-energy trivial states, 2015, arXiv:quantum-ph/1510.02082.
- [FV15] Joseph Fitzsimons and Thomas Vidick. A multiprover interactive proof system for the local Hamiltonian problem. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science, (ITCS '15)*, pages 103–112, 2015.
- [GHLS15] Sevag Gharibian, Yichen Huang, Zeph Landau, and Seung Woo Shin. Quantum Hamiltonian complexity. *Foundations and Trends in Theoretical Computer Science*, 10(3):159–282, 2015.
- [HNN13] Sean Hallgren, Daniel Nagaj, and Sandeep Narayanaswami. The local Hamiltonian problem on a line with eight states is QMA-complete. *Quantum Info. Comput.*, 13(9-10):721–750, September 2013.
- [Ji15] Zhengfeng Ji. Classical verification of quantum proofs, 2015, arXiv:quantum-ph/1505.07432.
- [KKR06] Julia Kempe, Alexei Kitaev, and Oded Regev. The complexity of the local Hamiltonian problem. *SIAM Journal on Computing*, 35(5):1070–1097, May 2006.
- [KR03] Julia Kempe and Oded Regev. 3-local Hamiltonian is QMA-complete. *Quantum Info. Comput.*, 3(3):258–264, 2003.
- [KSV02] A. Kitaev, A. Shen, and M. N. Vyalyi. *Classical and quantum computation*. Graduate studies in mathematics. American mathematical society, 2002.
- [NV15] Anand Natarajan and Thomas Vidick. Constant-soundness interactive proofs for local Hamiltonians, 2015, arXiv:quantum-ph/1512.02090.
- [Osb12] Tobias J Osborne. Hamiltonian complexity. *Reports on Progress in Physics*, 75(2):022001, 2012.
- [OT10] Roberto Oliveira and Barbara M. Terhal. The complexity of quantum spin systems on a two-dimensional square lattice. *Quantum Info. Comput.*, 8(10):900–924, 2010.
- [Raz98] Ran Raz. A parallel repetition theorem. *SIAM Journal on Computing*, 27(3):763–803, June 1998.
- [Raz05] Ran Raz. Quantum information and the PCP theorem. In *Proceedings of 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS '05)*, 2005.

A Proof of Lemma 4.4

Proof of Lemma 4.4. We remind here the notation from Section 3.3. The encoding of qubit k can be split into two parts: a GHZ-like state and copies of $|0\rangle$. (For the formal definition see Eqs. (1) and (2).) For the encoding of qubit k , let Q_k be the subset of provers that receive a share of the GHZ-like state and $\mathcal{S}_k \cong \mathbb{C}^{4^{|Q_k|}}$ its corresponding subspace. The projection over \mathcal{S}_k onto the codespace of the encoding of qubit k is

$$\Pi_k = \frac{1}{2} \left(\sum_{u,v \in \{0,1\}} |u^{|Q_k|}\rangle \langle v^{|Q_k|}| + \sum_{w,z \in \{2,3\}} |w^{|Q_k|}\rangle \langle z^{|Q_k|}| \right).$$

Let i be the qubit whose encoding was asked. Since the provers are dishonest and follow the same strategy, they all answer with the encoding of a qubit $j \neq i$.

We split the argument into two cases. If $Q_i \subset Q_j$ then the qudits that should contain the GHZ-like state on the encoding of i contain the reduced state of a bigger GHZ-like state and the density matrix of this reduced state is of the form

$$\rho_1 = \frac{|\alpha|^2}{2} (|0^{|Q_i|}\rangle \langle 0^{|Q_i|}| + |1^{|Q_i|}\rangle \langle 1^{|Q_i|}|) + \frac{|\beta|^2}{2} (|2^{|Q_i|}\rangle \langle 2^{|Q_i|}| + |3^{|Q_i|}\rangle \langle 3^{|Q_i|}|)$$

for some α and β , depending on the j -th qubit, with $|\alpha|^2 + |\beta|^2 = 1$. This case is depicted in Fig. 3a. The probability that this state projects onto Π_i is

$$\begin{aligned} \text{Tr}(\Pi_i \rho_1) &= \frac{|\alpha|^2}{2} (\text{Tr}(\Pi_i |0^{|Q_i|}\rangle \langle 0^{|Q_i|}|) + \text{Tr}(\Pi_i |1^{|Q_i|}\rangle \langle 1^{|Q_i|}|)) \\ &\quad + \frac{|\beta|^2}{2} (\text{Tr}(\Pi_i |2^{|Q_i|}\rangle \langle 2^{|Q_i|}|) + \text{Tr}(\Pi_i |3^{|Q_i|}\rangle \langle 3^{|Q_i|}|)) \\ &= \frac{|\alpha|^2}{2} \left(\frac{1}{2} + \frac{1}{2} \right) + \frac{|\beta|^2}{2} \left(\frac{1}{2} + \frac{1}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

where the second equality comes from the fact that $\|\Pi_i |i^{|Q_i|}\rangle\|^2 = \frac{1}{2}$ for $i \in \{0, 1, 2, 3\}$.

If $Q_i \not\subset Q_j$ then the set $Q_i \setminus Q_j$ is non-empty. Let $|Q_i \setminus Q_j| = r$ and $|Q_i \cap Q_j| = s$. The provers in $Q_i \setminus Q_j$ answer $|0\rangle$ and the remaining provers send either a GHZ-like state (when $Q_j \subset Q_i$) or a reduced state of a GHZ-like state (when $Q_j \not\subset Q_i$). These cases are depicted in Figs. 3b and 3c. Therefore, the answer from the provers has the form (up to some permutation of the qubits, but since the projection is symmetric, we can consider any arbitrary order)

$$\rho_2 = |0^r\rangle \langle 0^r| \otimes \sigma$$

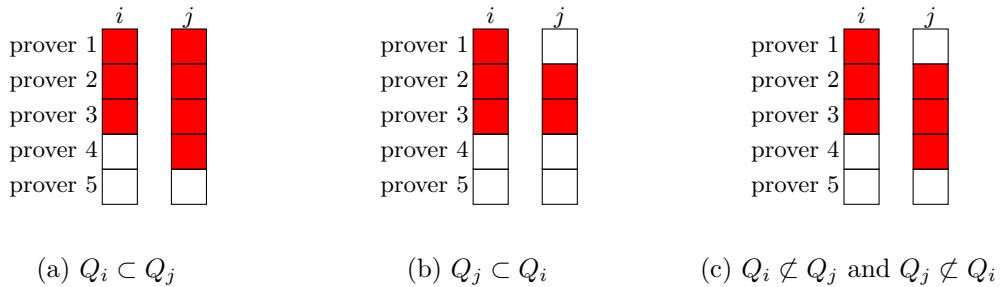


Figure 3: Examples of the cases for the expected vs. received encodings. The red cells correspond to shares of GHZ-like states and the white cells correspond to copies of $|0\rangle$.

for some density matrix σ on s qudits.

From the structure of Π_i and using that $|Q_i| = r + s$, we have that $\|\Pi_i |0^r\rangle \otimes |u^s\rangle\| = 0$ for $u \in \{1, 2, 3\}$. Let $\sigma_0 \stackrel{\text{def}}{=} |0^s\rangle\langle 0^s| \sigma |0^s\rangle\langle 0^s|$ be the projection of σ onto $|0^s\rangle\langle 0^s|$. It follows that the probability that the answered state projects onto Π_i is

$$\begin{aligned} \text{Tr}(\Pi_i \rho_2) &= \text{Tr}(\Pi_i |0^r\rangle\langle 0^r| \otimes \sigma) \\ &= \text{Tr}(\Pi_i |0^r\rangle\langle 0^r| \otimes \sigma_0) \\ &\leq \text{Tr}\left(\Pi_i |0^{r+s}\rangle\langle 0^{r+s}|\right) \\ &= \frac{1}{2}. \end{aligned}$$

Therefore, the answered state projects onto the codespace with probability at most $1/2$. \square